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Conformal structure of $\mathscr{I}^+$ and asymptotic symmetry I. Definitions and local theory

J Foster

School of Mathematical and Physical Sciences, University of Sussex, Falmer, Brighton, BN1 9QH, UK

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Abstract. The intuitive definition of asymptotic symmetry is compared with more formal definitions due to Tamburino and Winicour, and to Penrose, and within the context of the approach of Newman and Unti to asymptotically flat empty space-times these are shown to be equivalent. Natural interpretations of the transformations inherent in the work of Newman and Unti are given in terms of the conformal approach of Penrose.

1. Introduction

The concept of asymptotic symmetry first appeared in the work of Bondi *et al* (1962) and its generalisation to the non-axisymmetric case by Sachs (1962), where the BMS (Bondi–Metzner–Sachs) group appeared as a group of coordinate transformations which left invariant the asymptotic form of the metric tensor representing gravitational radiation from isolated sources in an asymptotically flat space-time. Newman and Unti (1962) also considered such radiation, but their definition of asymptotic flatness was less restrictive than that of Bondi and Sachs, this being due to the different approach.

Bondi and Sachs start with a form for the metric tensor and impose certain conditions on it to ensure flatness as a luminosity distance r tends to infinity. Among these conditions is one which ensures that in the limit the two-surfaces of constant r, in the null hypersurfaces around which their coordinate system is built, are spheres. Newman and Unti, on the other hand, express their condition for asymptotic flatness in terms of Ψ_0, one of the tetrad components of the Riemann tensor in the spin-coefficient formalism, and the asymptotic two-surfaces are not required to be spherical. Their specification is, in fact, part of the data for the initial-value problem. In the language of Penrose (1963, 1964, 1965), Bondi and Sachs require $\mathscr{I}^+$ to be homeomorphic to $\boldsymbol{R} \times \boldsymbol{S}^2$, but Newman and Unti admit more generality in the structure of $\mathscr{I}^+$.

It is of course arguable that $\mathscr{I}^+$ should be homeomorphic to $\boldsymbol{R} \times \boldsymbol{S}^2$ for the case of radiation from isolated sources (see Penrose 1965 and the proposed sequel to the present paper (to appear)) and in later papers Newman and his co-workers (e.g. Janis and Newman 1965, Newman and Penrose 1968) take it to be so. However, the original approach of Newman and Unti allows one to consider the question of asymptotic symmetry in cases of different global structures for $\mathscr{I}^+$, and to see how its structure affects the asymptotic symmetry group. This question was effectively raised in the closing remarks of the discussion in the original paper of Newman and Unti

(1962), and the proposed sequel will attempt to furnish the answer. The present paper contains some necessary preliminaries, and is largely concerned with clarifying the concept of an asymptotic symmetry transformation, and interpreting such transformations, and others introduced by Newman and Unti, in the conformal language of Penrose.

The definition of an asymptotic symmetry transformation used by Bondi and Sachs and by Newman and Unti was intuitive, being simply a coordinate transformation which preserves the asymptotic form of the metric tensor. Tamburino and Winicour (1966) have given a more formal definition, and Penrose (1963, 1964) has suggested a third; these last two are based on the conformal techniques of Penrose. The main purpose of this paper is to show that for the space–times admitted by the Newman–Unti condition for asymptotic flatness, these definitions are essentially equivalent, and to interpret the more general transformations of Newman and Unti in conformal geometric terms. It is therefore necessary to review that part of the work of Newman and Unti which has a bearing on asymptotic symmetry, and to translate it into the conformal language of Penrose.

The notation used is that of Newman and Unti, with the exception that Greek suffixes take the values 0, 1, 2, 3, lower case Latin suffixes the values 2, 3, and capitals the values 0, 2, 3.

2. Conformal aspects of the transformations of Newman and Unti

Newman and Unti use a coordinate and null tetrad system built around a family of null hypersurfaces: the coordinate $u = x^0$ labels these hypersurfaces, $r = x^1$ is an affine parameter along the null geodesics lying in the hypersurfaces, and the remaining coordinates x^i, $i = 2, 3$, serve to pick out these geodesics within each hypersurface. The hypersurfaces are required to be neither asymptotically plane nor cylindrical; this condition may be expressed in terms of the divergence, whose negative in the spin-coefficient formalism is ρ, and the complex shear σ of the null geodesics within the hypersurfaces:

$$\text{as } r \to \infty, \qquad \rho \neq 0 \neq \sigma, \qquad \rho^2 \neq \sigma\bar{\sigma}.$$

The condition of Newman and Unti for asymptotic flatness may be expressed as a condition on the tetrad component Ψ_0 of the Weyl tensor,

$$\Psi_0 = \Psi_0^0 r^{-5} + \mathrm{O}(r^{-6}), \tag{2.1}$$

together with a condition which they call *asymptotic smoothness*, which governs the behaviour of Ψ_0 under differentiation. Using this condition, and exploiting the spin-coefficient formalism, they are able to obtain an asymptotic solution of the empty-space field equations. From their expression for the asymptotic form of the components $g^{\mu\nu}$ of the metric tensor may be obtained the line element

$$\begin{aligned} \mathrm{d}s^2 = &-(a_{-1}r + a_0 + a_1 r^{-1} + \mathrm{O}(r^{-2}))\,\mathrm{d}u^2 + 2\,\mathrm{d}u\,\mathrm{d}r + P^{-2}(\delta_{ij}b_2^i + \mathrm{O}(r^{-1}))\,\mathrm{d}u\,\mathrm{d}x^j \\ &-\tfrac{1}{2}P^{-2}(r^2\delta_{ij} + \tfrac{1}{2}P^{-2}\,d_3^{kl}\delta_{ki}\delta_{lj}r + \sigma^0\bar{\sigma}^0\delta_{ij} + \mathrm{O}(r^{-1}))\,\mathrm{d}x^i\,\mathrm{d}x^j, \end{aligned} \tag{2.2}$$

where

$$a_{-1} = 2(\ln P)_{,0}, \qquad a_0 = -2P^2\nabla\bar{\nabla}\ln P, \qquad a_1 = -(\Psi_2^0 + \bar{\Psi}_2^0),$$

$$b_2^i = -(\xi^{0i}\bar{\omega}^0 + \bar{\xi}^{0i}\omega^0), \qquad d_3^{ij} = 2(\bar{\sigma}^0\xi^{0i}\xi^{0j} + \sigma^0\bar{\xi}^{0i}\bar{\xi}^{0j}),$$

and here

$$\xi^{02} = -\mathrm{i}\xi^{03} = P(u, x^i), \qquad \omega^0 = P\bar{\nabla}\sigma^0 - 2\sigma^0\bar{\nabla}P,$$
$$\nabla = \partial/\partial x^2 + \mathrm{i}\,\partial/\partial x^3, \qquad f_{,0} = \partial f/\partial u.$$

(A superscript zero indicates independence of r. The meaning of the order symbol is that of Newman and Unti (1962); included in it are terms which may often be calculated, but series expansions are taken only as far as is necessary for the present discussion.)

Apart from Ψ_2^0, which is the leading coefficient in the expansion of the tetrad component Ψ_2 of the Weyl tensor in inverse powers of r, the only quantities involved in this asymptotic form of the line element are σ^0 the leading coefficient in the expansion of the complex shear, and the function $P(u, x^i)$, together with their derivatives. (Here the function $P(u, x^i)$ is taken to be real-valued, whereas at this stage Newman and Unti have it complex-valued. It may be made real-valued by exploiting a still available tetrad transformation; Newman and Unti postpone this transformation until after considering the coordinate transformation induced by the introduction of a new family of hypersurfaces. It is more convenient to interchange these steps, so $P(u, x^i)$ is taken to be real-valued at this stage.)

The quantities $(\Psi_2^0 + \bar{\Psi}_2^0)$, σ^0 and $P(u, x^i)$ constitute part of the data for the initial-value problem. From the line element (2.2) it may be seen that the datum $P(u, x^i)$ effectively describes the geometry of the asymptotic two-surface which is the limit as $r \to \infty$ of the two-surface u, $r =$ constant. In some respects $P(u, x^i)$ is the most significant datum, since from a global point of view its prescription demands a consideration of the topological structure of the asymptotic two-surfaces, and these constitute the domain of the remaining initial data. This point will be taken up in the sequel to this paper.

In obtaining the asymptotic solution above, Newman and Unti make use of a number of tetrad and coordinate transformations to simplify the working, and a number of coordinate conditions are introduced; none of these changes the original family of null hypersurfaces. One of the more significant coordinate conditions is that of requiring that the coordinates x^i, regarded as coordinates on the two-surfaces u, $r =$ constant, be asymptotically isothermal; this is the origin of the function P.

They then go on to consider the effect of introducing a new coordinate system based on a new family of hypersurfaces $u' =$ constant, in which the same coordinate conditions hold, and calculate the transformation from the old to the new coordinate system. The infinitesimal version of this, $x^\mu \to x^\mu + \zeta^\mu(x^\nu)$, is given by

$$\begin{aligned} \zeta^0 &= \zeta^{00}(u, x^i), \\ \zeta^1 &= -\zeta^{00}{}_{,0}r + \mathrm{O}(1), \\ \zeta^i &= \zeta^{0i}(x^j) + \mathrm{O}(r^{-1}), \end{aligned} \tag{2.3}$$

where

$$\zeta^{02}{}_{,2} = \zeta^{03}{}_{,3}, \qquad \zeta^{02}{}_{,3} = -\zeta^{03}{}_{,2}. \tag{2.4}$$

The restrictions on ζ^{0i} reflect the asymptotically isothermal requirement. The function P transforms according to

$$P'(u + \zeta^0, x^i + \zeta^i) = (1 - \zeta^0_{,0} + \tfrac{1}{2}\zeta^i_{,i})P(u, x^i). \tag{2.5}$$

This then is the transformation which in a *general* way preserves the asymptotic form of the line element (2.2). Its finite form is

$$
\begin{aligned}
u' &= V_0(u, x^i) + \mathrm{O}(r^{-1}), \\
r' &= R_1(u, x^i) + \mathrm{O}(1), \\
x^{m'} &= Y_0^m(x^i) + \mathrm{O}(r^{-1}),
\end{aligned}
\tag{2.6}
$$

where $R_1 = (V_{0,0})^{-1}$, and (for transformations which are continuously derived from the identity)

$$
Y^2_{0,2} = Y^3_{0,3}, \qquad Y^2_{0,3} = -Y^3_{0,2}. \tag{2.7}
$$

In the finite form P transforms according to

$$
P'(V_0, Y_0^m) = (V_{0,0})^{-1}(\det(Y^i_{0,j}))^{1/2} P(u, x^i). \tag{2.8}
$$

For the remainder of this paper such a transformation will be referred to as an NU *transformation.* Newman and Unti make use of it to rid P of its u-dependence. The significance of this from the conformal point of view is given at the end of this section.

Asymptotic two-surfaces have been mentioned above, although a rigorous definition of such objects was not given. An effective way of dealing with them is to use the conformal techniques of Penrose, and to realise them as slices of the null hypersurface $\mathscr{I}^+$ which represents future null infinity. The limit as $r \to \infty$ of the transformation (2.6) is in fact a conformal mapping of $\mathscr{I}^+$ onto itself, and may be interpreted as a mapping between two families of asymptotic two-surfaces in which each member of one family is mapped onto a member of the other, as will now be shown.

The technique developed by Penrose (1964, 1965) (see also Walker 1972) for the discussion of asymptotic properties of space–time is by now well known, and familiarity with the basic ideas will be assumed. The space–times discussed in detail by Penrose have the property of *asymptotic simplicity*, a property which comprises a number of regularity conditions and a condition of a topological nature. This last requires that every null geodesic in the unphysical space–time $\mathscr{M}$ contains, if maximally extended, two distinct points on the boundary $\mathscr{I}$, and Penrose (1965) has shown that in the cases where $\mathscr{I}$ is null this implies that the two subsets $\mathscr{I}^+$ and $\mathscr{I}^-$ of $\mathscr{I}$ are both homeomorphic to $\boldsymbol{R} \times \boldsymbol{S}^2$. In this paper the last requirement is dropped, so as to admit the wider class of space–times which the approach of Newman and Unti allows.

These space–times satisfy the empty-space field equations with zero cosmological constant, so $\mathscr{I}$ is null. The parameter r is an affine parameter along null geodesics and it is assumed that r increases into the future, so that $\mathscr{I}^+$ is reached as $r \to \infty$. If r is replaced by a new coordinate $l = 1/r$, then the unphysical space–time $\mathscr{M}$ with line element $\mathrm{d}s^2$ may be obtained from the physical space–time $\tilde{\mathscr{M}}$ with line element $\mathrm{d}\tilde{s}^2$ by a conformal transformation $\mathrm{d}s^2 = l^2\,\mathrm{d}\tilde{s}^2$, and $\mathscr{I}^+$ is given by $l = 0$. The line element of $\mathscr{M}$ is then

$$
\begin{aligned}
\mathrm{d}s^2 = {} & -(a_{-1}l + a_0 l^2 + a_1 l^3 + \mathrm{O}(l^4))\,\mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}l + P^{-2}(\delta_{ij} b^i_2 l^2 + \mathrm{O}(l^3))\,\mathrm{d}u\,\mathrm{d}x^j \\
& - \tfrac{1}{2}P^{-2}(\delta_{ij} + \tfrac{1}{2}P^{-2}\, d_3^{kl}\delta_{ki}\delta_{lj} l + \sigma^0\bar{\sigma}^0 l^2 \delta_{ij} + \mathrm{O}(l^3))\,\mathrm{d}x^i\,\mathrm{d}x^j.
\end{aligned}
\tag{2.9}
$$

Asymptotic two-surfaces are then the slices of $\mathscr{I}^+$ given by $u = \text{constant}$. On putting $l = 0$ and $u = \text{constant}$ in the line element (2.9) one gets

$$
\mathrm{d}s_0^2 = \tfrac{1}{2}(P(u, x^k))^{-2}\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \tag{2.10}
$$

as the line element of the two-surface u = constant, and x^i are isothermal coordinates for each such two-surface. This is also the line element of $\mathscr{I}^+$, since dropping the condition u = constant and simply putting $l = 0$ in the expression (2.9) yields (2.10).

The stage has now been reached where it is possible to give a more natural formulation of an NU transformation (in its infinitesimal form (2.3)). Replacing $x^1 = r$ by the new coordinate $x^1 = l = 1/r$, it takes the form $x^\mu \to x^\mu + \zeta^\mu(x^\nu)$ where

$$\zeta^0 = \zeta^{00}(u, x^i), \qquad \zeta^1 = \zeta^{00}{}_{,0}l + O(l^2), \qquad \zeta^i = \zeta^{0i}(x^j) + O(l), \tag{2.11}$$

and ζ^{0i} satisfy equation (2.4). The second of the equations in (2.11) shows that $\mathscr{I}^+$ is left invariant by this transformation (i.e. $l = 0 \Leftrightarrow l' = 0$), and it therefore may be regarded as a mapping of $\mathscr{I}^+$ onto itself, whose infinitesimal generators are given by (on dropping the superscript zero)

$$\zeta^0 = \zeta^0(u, x^i), \qquad \zeta^i = \zeta^i(x^j), \tag{2.12}$$

where

$$\zeta^2{}_{,2} = \zeta^3{}_{,3}, \qquad \zeta^2{}_{,3} = -\zeta^3{}_{,2}. \tag{2.13}$$

No information is in fact lost by adopting this point of view because the terms included in the order symbols in the relations (2.3) are determined by ζ^{00} and ζ^{0i} (see Newman and Unti 1962).

Infinitesimal generators of the form (2.12) satisfying (2.13) are in fact precisely the generators of conformal transformations of $\mathscr{I}^+$ onto itself. That is, they are the solutions of

$$g_{AB,C}\zeta^C + g_{DB}\zeta^D{}_{,A} + g_{AD}\zeta^D{}_{,B} - \lambda g_{AB} = 0, \tag{2.14}$$

$(A, B, C, D = 0, 2, 3)$. For on taking

$$[g_{AB}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}P^{-2} & 0 \\ 0 & 0 & \frac{1}{2}P^{-2} \end{bmatrix},$$

the equations (2.14) become

$$\zeta^2{}_{,0} = \zeta^3{}_{,0} = 0,$$
$$\lambda = 2\zeta^2{}_{,2} - 2P^{-1}P_{,A}\zeta^A = 2\zeta^3{}_{,3} - 2P^{-1}P_{,A}\zeta^A,$$
$$\zeta^3{}_{,2} + \zeta^2{}_{,3} = 0,$$

from which may readily be deduced the form (2.12) of the generators and the conditions (2.13) on their derivatives, together with the expression

$$\lambda = \zeta^i{}_{,i} - 2P^{-1}P_{,A}\zeta^A \tag{2.15}$$

for λ.

Thus an NU transformation has a natural interpretation as a conformal transformation of $\mathscr{I}^+$ onto itself. It does, however, require a different point of view. The transformation as derived by Newman and Unti is a *coordinate* transformation; in the interpretation now given to it, it is a *point* transformation of $\mathscr{I}^+$ into itself. The finite version takes the form

$$u' = V(u, x^i), \qquad x'^m = Y^m(x^i), \tag{2.16}$$

where

$$Y^2{}_{,2} = Y^3{}_{,3}, \qquad Y^2{}_{,3} = -Y^3{}_{,2}. \tag{2.17}$$

These equations are just the same as the limit of equations (2.6), but the interpretation now is that the point on $\mathscr{I}^+$ with coordinates (u, x^i) is mapped into the point with coordinates $(V(u, x^i), Y^m(x^i))$. (The difference in the positions of the primes on the coordinates x^m in equations (2.6) and (2.16) is one of the niceties of the kernel-index notation which enables one to distinguish a point from a coordinate transformation.)

The distance between two neighbouring points with coordinates (u, x^i) and $(u + du, x^i + dx^i)$ is given by

$$ds_0^2 = \tfrac{1}{2}(P(u, x^k))^{-2}\delta_{ij}\, dx^i\, dx^j, \tag{2.18}$$

and these get mapped into two neighbouring points whose distance apart is given by

$$ds_0'^2 = \tfrac{1}{2}(P(V(u, x^k), Y^m(x^k)))^{-2} \det(Y^m{}_{,n}(x^k))\delta_{ij}\, dx^i\, dx^j, \tag{2.19}$$

so the mapping is indeed conformal.

If the two points lie on the same two-surface $u = \text{constant} = u_0$, say, then $du = 0$, the relations (2.18) and (2.19) are unchanged, and one sees that the two-surface $u = u_0$ is mapped conformally onto the two-surface given by $u = V(u_0, x^i)$. In this way a conformal transformation of $\mathscr{I}^+$ induces conformal transformations between members of families of asymptotic two-surfaces.

Finally, one other aspect of the work of Newman and Unti has a conformal interpretation. Because $\mathscr{I}^+$ is null it contains null geodesics or *generators* (a conformally invariant concept); they are given by $x^i = \text{constant}$ and u is a parameter along them. If one calculates their divergence (using Christoffel symbols of $\mathscr{M}$ and then going to $\mathscr{I}^+$), one finds that it is $(\ln P)_{,0}$; on the other hand their shear is zero. The vanishing of the shear is a conformally invariant property, but the vanishing or non-vanishing of the divergence is not. Thus in making use of an NU transformation to rid P of its u-dependence, Newman and Unti are effectively introducing a new conformal factor $l' = 1/r'$ relating the physical space-time to an unphysical one in which the generators of $\mathscr{I}^+$ are non-diverging.

3. Asymptotic symmetry

Penrose (1964) has pointed out that by using his conformal technique it is possible to discuss asymptotic symmetries. Any motion which takes the physical space–time $\tilde{\mathscr{M}}$ into itself produces a conformal motion of the unphysical space–time $\mathscr{M}$ which induces a conformal motion of $\mathscr{I}^+$. Even if $\tilde{\mathscr{M}}$ does not have exact symmetries, conformal motions of $\mathscr{I}^+$ may persist, and this provides a tentative definition of an asymptotic symmetry transformation. When viewed as a mapping of $\mathscr{I}^+$ onto itself an NU transformation is an asymptotic symmetry transformation according to this definition.

However, under such an infinitesimal transformation, the function $P(u, x^i)$ transforms according to equation (2.8), and its *exact* form is not preserved, nor is the asymptotic form of the line element (2.2). The transformation therefore lacks the essential property which one intuitively feels is the correct defining property for an asymptotic symmetry transformation.

The reason that this definition is unsatisfactory for the space–times considered here is that $\mathscr{I}^+$ is null; its singular metric endows it with insufficient structure. In order to get a satisfactory conformal definition one needs a way of restricting the conformal transformations of $\mathscr{I}^+$ to those whose generators are the limits of generators of coordinate transformations preserving P. This requires ζ^0 to satisfy

$$\zeta^0{}_{,0}-\tfrac{1}{2}\zeta^i{}_{,i}+P^{-1}(P_{,0}\zeta^0+P_{,i}\zeta^i)=0. \tag{3.1}$$

This follows from equation (2.5) by imposing the condition $P'(u,x^i)=P(u,x^i)$ and expanding its left-hand side in a Taylor series up to first-order quantities.

The definition of an asymptotic symmetry transformation given by Tamburino and Winicour (1966) effectively does this, and the following is an outline of their argument.

If the physical space–time $\tilde{\mathcal{M}}$ has exact symmetries, then there exists a Killing vector field ζ^μ satisfying

$$\tilde{g}_{\mu\nu,\rho}\zeta^\rho+\tilde{g}_{\rho\nu}\zeta^\rho{}_{,\mu}+\tilde{g}_{\mu\rho}\zeta^\rho{}_{,\nu}=0, \tag{3.2}$$

which may be written in terms of the unphysical metric tensor $g_{\mu\nu}=\Omega^2\tilde{g}_{\mu\nu}$ as

$$g_{\mu\nu,\rho}+g_{\rho\nu}\zeta^\rho{}_{,\mu}+g_{\mu\rho}\zeta^\rho{}_{,\nu}-2\Omega^{-1}\Omega_{,\rho}\zeta^\rho g_{\mu\nu}=0, \tag{3.3}$$

and in particular this equation holds at $\mathscr{I}^+$. (Hence $\Omega_{,\rho}\zeta^\rho=0$ at $\mathscr{I}^+$, indicating that ζ^μ is tangent to $\mathscr{I}^+$.) Even if $\tilde{\mathcal{M}}$ does not have exact symmetries there may exist quantities ζ^μ satisfying equation (3.3) at $\mathscr{I}^+$, and in this way an *asymptotic Killing vector field* may be defined.

As remarked above, an asymptotic Killing vector is tangent to $\mathscr{I}^+$, so under the transformation of $\mathcal{M}$ generated by it $\mathscr{I}^+$ is mapped onto itself. Thus one can give a precise definition of an asymptotic symmetry transformation as a mapping of $\mathscr{I}^+$ onto itself generated by an asymptotic Killing vector field. This essentially is the definition given by Tamburino and Winicour.

For the space–times under consideration $\Omega=l=x^1$, and the indeterminate factor $\Omega^{-1}\Omega_{,\rho}\zeta^\rho$ may be evaluated by l'Hôpital's rule. The equation to be satisfied becomes

$$[g_{\mu\nu,\rho}\zeta^\rho+g_{\rho\nu}\zeta^\rho{}_{,\mu}+g_{\mu\rho}\zeta^\rho{}_{,\nu}-2\zeta^1{}_{,1}g_{\mu\nu}]_{\mathscr{I}^+}=0. \tag{3.4}$$

Since $\zeta^1=0$ at $\mathscr{I}^+$, this last equation yields the following information valid at $\mathscr{I}^+$:

$$\zeta^1{}_{,0}=0, \tag{3.5a}$$

$$\zeta^0{}_{,0}=\zeta^1{}_{,1}, \tag{3.5b}$$

$$\zeta^i{}_{,0}=-2P^2\delta^{ij}\zeta^1{}_{,j}, \tag{3.5c}$$

$$\zeta^0{}_{,1}=0, \tag{3.5d}$$

$$\zeta^i{}_{,1}=-2P^2\delta^{ij}\zeta^0{}_{,j}, \tag{3.5e}$$

$$P^{-1}(P_{,0}\zeta^0+P_{,i}\zeta^i)-\zeta^2{}_{,2}+\zeta^1{}_{,1}=0, \tag{3.5f}$$

$$P^{-1}(P_{,0}\zeta^0+P_{,i}\zeta^i)-\zeta^3{}_{,3}+\zeta^1{}_{,1}=0, \tag{3.5g}$$

$$\zeta^3{}_{,2}+\zeta^2{}_{,3}=0. \tag{3.5h}$$

Equation (3.5d) implies that $\zeta^0=\zeta^{00}(u,x^i)+O(l^2)$; equation (3.5$b$) then implies that $\zeta^1=\zeta^{00}{}_{,0}l+O(l^2)$ off $\mathscr{I}^+$, and (3.5a) is satisfied. Thus $\zeta^1{}_{,j}=0$ on $\mathscr{I}^+$, and (3.5c) implies $\zeta^i{}_{,0}=0$ on $\mathscr{I}^+$. Equation (3.5e) then implies that $\zeta^i=\zeta^{0i}(x^j)-2P^2\delta^{ij}\zeta^{00}{}_{,j}l+O(l^2)$ off $\mathscr{I}^+$. So the mapping of $\mathscr{I}^+$ onto itself is given by generators whose components have the form (on dropping the superscript zero)

$$\zeta^0=\zeta^0(u,x^i),\qquad \zeta^i=\zeta^i(x^j).$$

The remaining equations (3.5*f*, *g*, *h*) yield relations between their derivatives. Equation (3.5*h*) and the difference of the equations (3.5*f*) and (3.5*g*) reproduce the relations (2.13). So the transformation generated is a conformal transformation of $\mathscr{I}^+$ onto itself. However, it is not a general one, for there still remains the sum of equations (3.5*f*) and (3.5*g*), which reduces to equation (3.1), and in this way the conformal transformations of $\mathscr{I}^+$ are restricted as required.

Penrose (1964) has indicated an alternative and more geometric way of restricting the conformal transformations of $\mathscr{I}^+$ in order to obtain a more sensible definition of an asymptotic symmetry transformation. A conformal transformation induces a mapping of the tangent space at a point onto the tangent space at the image of that point in which angles are preserved. Because $\mathscr{I}^+$ is null there exist zero angles between non-proportional vectors; these occur when the two vectors are coplanar with the tangent to a generator of $\mathscr{I}^+$. Penrose calls these *null angles.* Although two null angles are both numerically zero, it nevertheless makes sense to say that one is larger than the other if it includes the other as a part, and Penrose has shown that one may use this fact to introduce a concept of inequality between null angles. The conformal geometry of $\mathscr{I}^+$ may then be strengthened by requiring conformal transformations to preserve null angles also. This in fact provides the necessary restriction on conformal transformations to make them asymptotic symmetry transformations, as the following shows.

Penrose's procedure for comparing null angles is quite sophisticated, and may be replaced in the present context by a procedure which allows one actually to measure the size of a null angle.

Consider then the null angle formed by a pair of tangents to $\mathscr{I}^+$ which are coplanar with the tangent vector to a generator of $\mathscr{I}^+$ at a point O with coordinates (u, x^i). If one identifies a small region of the tangent space at O with a small region of $\mathscr{I}^+$ about O, then the two vectors intersect a neighbouring generator in the points P_1 and P_2 with coordinates $(u_1, x^i + dx^i)$ and $(u_2, x^i + dx^i)$ respectively, say. (See figure 1.)

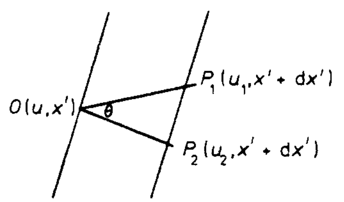

Figure 1.

Let d be given by

$$\mathrm{OP}_1^2 = \mathrm{OP}_2^2 = d^2 = \tfrac{1}{2}(P(u, x^i))^{-2}\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j,$$

and define the *size* of the null angle $\angle \mathrm{P}_1\mathrm{OP}_2$ to be

$$\theta = \lim_{\mathrm{d}x^i \to 0} |u_1 - u_2|/d.$$

This definition is in fact conformally invariant in the sense that if one makes an NU transformation and uses the corresponding new conformal factor $\Omega' = l' = 1/r'$ to obtain the unphysical space–time, then the size of the null angle is not changed.

Under an infinitesimal conformal transformation of $\mathscr{I}^+$ given by $x^A \to x^A + \zeta^A$ $(A = 0, 2, 3)$, the quantities $(u_1 - u_2)$ and d transform according to

$$u_1 - u_2 \to u_1 - u_2 + \zeta_1^0 - \zeta_2^0,$$
$$d \to [1 - (P(u, x^i))^{-1}(P_{,0}\zeta^0 + P_{,i}\zeta^i) + \tfrac{1}{2}\zeta^i{}_{,i}]d.$$

The transformation for d is a consequence of equation (2.18) and the infinitesimal form of equation (2.19).

Hence the null angle is mapped into one whose size is

$$\lim_{dx^i \to 0}\left[\frac{|u_1 - u_2|}{d}\left(1 + \frac{\zeta_1^0 - \zeta_2^0}{u_1 - u_2} + P^{-1}(P_{,0}\zeta^0 + P_{,i}\zeta^i) - \tfrac{1}{2}\zeta^i{}_{,i}\right)\right]$$
$$= [1 + \zeta^0{}_{,0} + P^{-1}(P_{,0}\zeta^0 + P_{,i}\zeta^i) - \tfrac{1}{2}\zeta^i{}_{,i}]\theta.$$

So null angles are preserved if and only if equation (3.1) is satisfied, and it is seen that asymptotic symmetry transformations are precisely those conformal transformations which preserve null angles.

4. Conclusions

As has been demonstrated above, the definitions of asymptotic symmetry transformations proposed by Tamburino and Winicour and by Penrose are equivalent, and agree with the intuitive definition, at least as far as the class of space–times admitted by the approach of Newman and Unti to asymptotically flat space–times are concerned. The NU transformations preserving the asymptotic form of the metric tensor in a *general* way are just the conformal transformations of $\mathscr{I}^+$ onto itself, while those which preserve the asymptotic form *exactly* are conformal transformations of $\mathscr{I}^+$ which in addition preserve null angles.

It has also been shown that the use of an NU transformation by Newman and Unti to rid the function $P(u, x^i)$ of its u-dependence may be interpreted as choosing a conformal factor which yields a form for $\mathscr{I}^+$ with its generators non-diverging.

As far as practical calculations of asymptotic symmetry transformations are concerned, the definition of Tamburino and Winicour is the easiest with which to work. Their definition was used in the author's calculation of the asymptotic symmetry groups of the DS-spaces of Robinson and Trautman (see Foster 1969).

No attention has been paid in the present paper to the question of the existence of (non-trivial) asymptotic symmetry transformations, nor to the relevance that the global structure of $\mathscr{I}^+$ has to this question. These matters will be considered in the proposed sequel to the present paper.

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